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POSITIVE SOLUTION OF LIGHTHILL-TYPE EQUATION IN $[0, \infty)$

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We study the solvability of Lighthill-type integral equations

$$u(t) + \lambda \int_0^t (t-s)^{-\alpha} s^{\beta} u^{\gamma}(s) ds = \mu, \ \ 0 \le t < \infty. \tag{1}$$

THEOREM 1. *Under conditions $\lambda > 0$, $\mu > 0$, $0 < \alpha < 1$, $\beta \ge 0$, $\gamma \ge 1$, equation (1) has a unique solution $u_\star \in C[0,\infty)$ such that $0 < u_\star(t) < \mu$ for $t > 0$, $u_\star(0) = \mu$.*

For the proof of Theorem 1 we need to study a more general class of equations

$$u(t) + \int_0^t (t-s)^{-\alpha} g(t,s,u(s)) ds = f(t), \ \ 0 \le t < \infty. \tag{2}$$

THEOREM 2. *Let $0 < \alpha < 1$. Assume that $f \in C[0,\infty)$, $f(t) > 0$ for $t > 0$ (so $f(0) \ge 0$) and $t^\alpha f(t)$ is monotonic increasing for $t \ge 0$. Assume that $g(t,s,u) \ge 0$ is continuous, monotonic decreasing w.r.t. t and monotonic increasing w.r.t. u for $0 \le t < \infty$, $0 \le s \le t$, $0 \le u \le f(s)$, satisfies the Lipschitz condition w.r.t. u for $0 \le t < T$, $0 \le s \le t$, $0 \le u \le f(s)$, $\forall T > 0$, and $g(t,s,0) \equiv 0$. Then equation (2) has a unique solution $u_\star \in C[0,\infty)$ such that $0 \le u_\star(t) \le f(t)$ for $t \ge 0$.*

THEOREM 3. *Let conditions of Theorem 2 be fulfilled and let f be locally Hölder continuous of degree $\rho \in (\alpha, 1]$. Let also g be w.r.t. t and s locally Hölder continuous of the same degree, and $g(t,s,u) > 0$ for $t > 0$, $0 < s < t$, $0 < u < f(s)$. Then strict inequalities $0 < u_\star(t) < f(t)$, $0 < t < \infty$, hold for the solution $u_\star$ of equation (2); clearly $u_\star(0) = f(0)$. (The meaning of local Hölder continuity will be given in the talk.)*

Theorem 1 is a consequence of Theorems 2 and 3 for $g(t,s,u) = \lambda s^\beta u^\gamma$, $f(t) \equiv \mu$. Numerical solution of equations (1) and (2) will be commented.

CONJECTURE. *Under conditions of Theorem 1, $u_\star(t) \to 0$ monotonically as $t \to \infty$.*
The conjecture is true for $\beta = 0$ but we had no success trying to prove it for $\beta > 0$.

The unique solvability (but not the positiveness of the solution) of equations of the form $u(t) + \int_0^t (t-s)^{-\alpha} s^\beta g(u(s)) ds = f(t)$, $0 \le t < \infty$, with $\beta > 0$, has been studied by Lisbon colleagues (Diogo's talk). Theorem 2 is in a good intercourse with their result.