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Superpositional graphs

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Abstract. The class of superpositional graphs, which is defined as the set of graphs whose elements are generated by superposition, is a subclass of binary graphs. The properties of superpositional graphs are investigated, and necessary and sufficient conditions found for a binary graph to be a superpositional graph.

1. Preliminaries and motivation

Let us denote $B = \{0, 1\}$. A *Boolean function* is a mapping $f : B^n \to B$. In this section we are using two different formalisms for representing Boolean functions: propositional formulae and decision diagrams.

Let $x_1, \ldots, x_n$ be propositional variables. Literals for variable x are x and $\bar{x}$. We denote by $var(l)$ the variable of literal l.

Definition 1. A *propositional formula* on the basis $\{\vee, \&, \neg\}$ is defined inductively as follows:
1° Every literal is a propositional formula.
2° If P and R are propositional formulae, then $(P \& R)$ and $(P \vee R)$ are propositional formulae.

The priority of Boolean connectives is $\vee, \&, \neg$ (the highest priority). We allow omitting parentheses if there is no confusion in determining the structure of subformulae. We denote the fact, that $F(\alpha_1, \ldots, \alpha_n) = 1$ for $\alpha = (\alpha_1, \ldots, \alpha_n) \in B^n$, by $\alpha \vdash F$.

Definition 2. A *binary graph* is an oriented acyclic connected graph with a root and two terminals (sinks) — 0 and 1. Every internal node v has

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two sucessors: $high(v)$ and $low(v)$. An edge $a \to b$ is a 0-*edge* (1-*edge*) if $low(a) = b$ ($high(a) = b$).

Definition 3. A *path* from a node u to a node v ($u \rightsquigarrow v$) is a sequence of nodes $z_0, \ldots, z_k$, where $z_0 = u$, $z_k = v$ and for each $0 \leq i < k$, $z_{i+1} = low(z_i)$ or $z_{i+1} = high(z_i)$.

Definition 4. A *binary decision diagram* (BDD) over a set of variables $\{x_1, \ldots, x_n\}$ is a binary graph, where every internal node v of G is labelled by a literal l, where $var(l) \in \{x_1, \ldots, x_n\}$.

Let D be a binary decision diagram with variables $x_1, \ldots, x_n$. Every assignment $\alpha \in B^n$ activates a path $p(\alpha) = p_1, \ldots, p_k$ in D from the root to a terminal node: if $\alpha \vdash label(v_i)$, then $v_{i+1} = high(v_i)$ else $v_{i+1} = low(v_i)$. A Boolean function $f_D(x_1, \ldots, x_n)$, represented by D, is defined as follows: $f(\alpha) = 1$ if and only if the path activated by α points to the terminal node 1.

Binary decision diagrams were first introduced by Lee [5] as a data structure for representing Boolean functions. They were further popularized by Akers [1] and Bryant [2]. There are many monographs and textbooks about BDDs, for example [3].

Definition 5. Let G and E be two binary graphs. A *superposition* of E into G instead of an internal node v ($G_{v \leftarrow E}$) is a graph, which is obtained by deleting v from G and redirecting all edges, pointing to v, to the root of E, all edges of E pointing to the terminal node 1 to the node $high(v)$ and all edges pointing to the terminal node 0 to the node $low(v)$.

Let A, C and D be binary graphs, whose descriptions are shown in Figure 1.

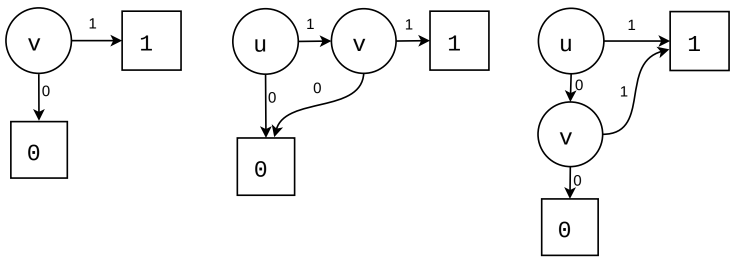

FIGURE 1. Binary graphs A, C and D.

Definition 6. The *class of superpositional graphs* (SPG) is defined inductively as follows:

1° The graph $A \in SPG$.

2° If $G \in SPG$ and v is an internal node of G, then $G_{v \leftarrow C} \in SPG$ and $G_{v \leftarrow D} \in SPG$.

Note that $C = A_{v \leftarrow C} \in SPG$ and $D = A_{v \leftarrow D} \in SPG$. We say that $v \leftarrow C$ and $v \leftarrow D$ are *elementary* superpositions.

Definition 7. A *structurally synthesized binary decision diagram* (SSBDD) for a formula F, $D(F)$, is a superpositional graph, defined inductively according to the structure of F:

1° If F is a literal l, then $D(F)$ is the graph A, where the root of A is labelled by l.

2° If $F = P\&R$, then $D(F)$ is the graph $C_{u \leftarrow D(P), v \leftarrow D(R)}$.

3° If $F = P \vee R$, then $D(F)$ is the graph $D_{u \leftarrow D(P), v \leftarrow D(R)}$.

Binary graphs A, C and D in Figure 1 are SSBDDs for the formulae v, $u\&v$ and $u \vee v$. The formula

$$a\&((((b \vee c)\&d) \vee e)\&f)$$

has the SSBDD, shown in Figure 2.

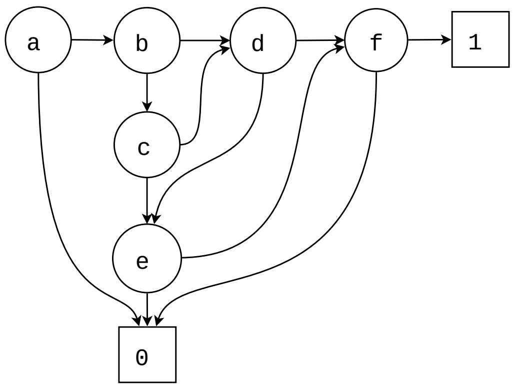

FIGURE 2. The SSBDD for the formula $a\&((((b \vee c)\&d) \vee e)\&f)$.

SSBDDs were first introduced by Ubar [6] and further developed in his doctoral thesis [7]. One of the fastest fault simulators in the world is based on structurally synthesized binary decision diagrams (see [8], [9]).

By standard conventions we direct in the figures 1-edges from left to right and 0-edges from up to down. In this case we can omit the labels 0 and 1 from the edges. The notions of an activated path and a Boolean function represented by the SSBDD are similar to the case of BDDs.

The following theorem is proven in [4].

Theorem 1. *A propositional formula F and its SSBDD $D(F)$ represent the same Boolean function.*

It is easy to see that superpositional graphs are a special case of binary graphs: the initial graph A is a binary graph and the superposition is preserving all properties in the definition of binary graphs. SSBDDs are based on superpositional graphs, alike BDDs are based on binary graphs. Many properties of SSBDDs are depending on the properties of superpositional graphs (see [4]). One of the problems, important for many applications, is: *given a BDD, is it a SSBDD*?

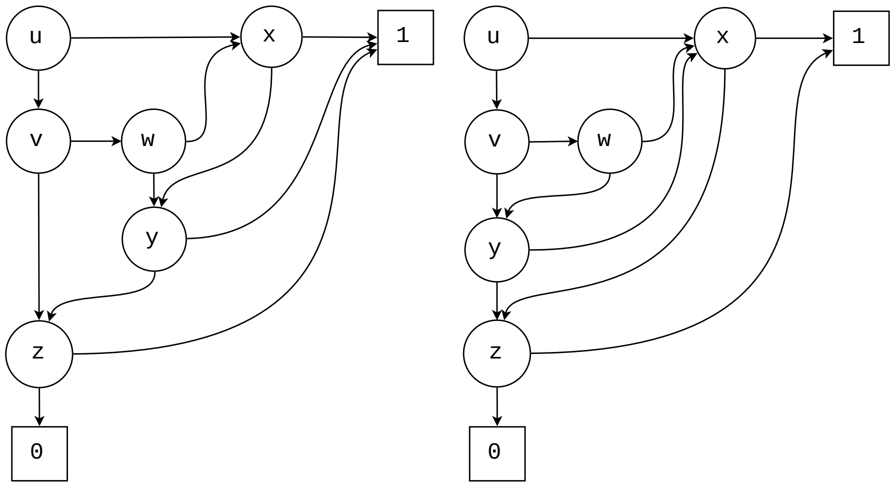

FIGURE 3. Two decision diagrams.

In Figure 3 there are two decision diagrams. Are they SSBDDs or not? It is obvious that both of them are BDDs, because they are both binary graphs. To determine, if the underlying graph is superpositional, we have to try to generate these graphs according to Definition 6. This means that we should check all possible sequences of elementary superpositions of length $n-1$, where n is the number of internal nodes of the binary graph. This method is obviously infeasible. A better idea would be to find necessary and sufficient graph-theoretical properties for a binary graph to be a superpositional graph. The goal of this paper is to find just such conditions.

2. Properties of superpositional graphs

Theorem 2. *If $G, H \in SPG$ and v is an internal node of G, then $G_{v\leftarrow H} \in SPG$ (the class of superpositional graphs is closed under superposition).*

Proof. By assumption $H \in SPG$ and $G \in SPG$ can be generated from the graph A by some sequence of elementary superpositions. To show that $G_{v\leftarrow H} \in SPG$ we generate graph G using elementary superpositions and perform the same sequence of elementary superpositions in the internal node v, which were implemented to get the graph H from the initial graph A. It is easy to see that the resulting graph $G_{v\leftarrow H}$ is in fact a superpositional graph. $\square$

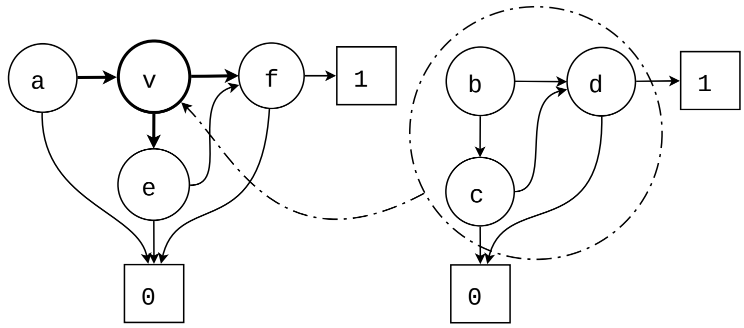

FIGURE 4. Superpositional graphs G and E before superposition.

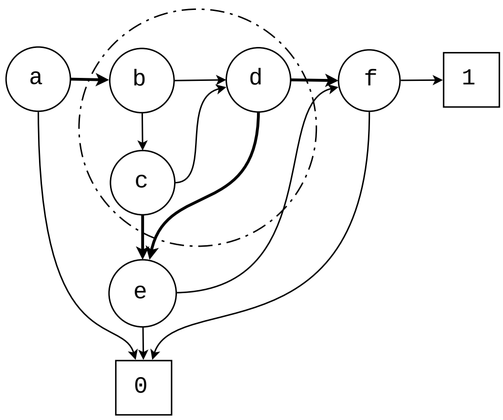

FIGURE 5. The graph $G_{v \leftarrow E}$.

Theorem 2 allows us to give an alternative definition of superpositional graphs, which is more convenient to use in the following proofs.

Definition 8. The *class SPG* of graphs is defined inductively as follows:
1° The graphs $A, C, D \in SPG$.
2° If $G, H \in SPG$ and v is an internal node of G, then $G_{v \leftarrow H} \in SPG$.

The example in Figures 4 and 5 illustrates the process of finding the superposition $G_{v \leftarrow E}$.

We will prove Theorems 3, 4 and 5 using induction by the alternative definition of superpositional graphs. The basis of induction is obvious in all cases: all the properties hold trivially for the graphs A, C and D.

Definition 9. A *0-path* (*1-path*) from a node u to a node v is a path, which contains only 0-edges (1-edges).

Theorem 3. *Let $G \in SPG$. Then for every internal node u there exist a 0-path $u \rightsquigarrow 0$ and a 1-path $u \rightsquigarrow 1$.*

Proof. Let $S = G_{v \leftarrow E}$. By induction hypothesis there exists a 0-path $u \rightsquigarrow 0$ in the graph G. If v does not belong to that 0-path $u \rightsquigarrow 0$ in G, then that 0-path remains unchanged in $G_{v \leftarrow E}$. If v is on the path, then we substitute v with the 0-path $root(E) \rightsquigarrow 0$, which exists by induction hypothesis. The proof for 1-paths is analogous. □

Definition 10. We say that a binary graph G is *homogenous* if only one type of edges enters into every node $v \in V(G)$.

Theorem 4. *Every superpositional graph is homogenous.*

Proof. Let $S = G_{v \leftarrow E}$. By induction hypothesis the graphs $G, E \in SPG$ are homogenous. By the execution of superposition $G_{v \leftarrow E}$, exiting edges of the graph E were redirected and edges pointing to the node v were redirected to the root of E. Exiting 0-edges of the graph E were redirected to the node $low(v)$, and since $low(v)$ had at least one 0-edge in G (the one from v), it remains homogenous after the substitution. The same argument works for $high(v)$ as well. □

Definition 11. A binary graph G is *traceable* if there exists a path through all internal nodes of G (a Hamiltonian path).

A binary graph is acyclic, therefore if a Hamiltonian path exists, then it is unique.

Theorem 5. *Every superpositional graph is traceable.*

Proof. Let $S = G_{v \leftarrow E}$. By induction hypothesis there exists a Hamiltonian path $u_1 \rightsquigarrow u_k$ in the graph G and a Hamiltonian path $w_1 \rightsquigarrow w_l$ in the graph E. The path $u_1 \rightsquigarrow u_k$ must include the node v, let $u_i = v$. By the definition of superposition we will obtain $high(w_l) = high(u_i)$, $low(w_l) = low(u_i)$ and $w_1 = high(u_{i-1})$ or $w_1 = low(u_{i-1})$. Therefore the Hamiltonian path in the graph S is $u_1, \ldots, u_{i-1}, w_1, \ldots, w_l, u_{i+1}, \ldots, u_k$. □

Theorem 5 gives a canonical enumeration of the nodes of a superpositional graph. Given the canonical enumerations of $G, H \in SPG$, we can test the existence of an isomorphism between G and H in time $O(n)$ (we must check endpoints of all edges, and there are $2n$ edges in a binary graph with n internal nodes).

Definition 12. We say that a binary graph G has the *triangle property* if for every three internal nodes x, y and z the existence of a 1-path $x \rightsquigarrow y$ and a 0-path $x \rightsquigarrow z$ implies the existence of either a 1-path $z \rightsquigarrow y$ or a 0-path $y \rightsquigarrow z$.

Theorem 6. *Every superpositional graph has the triangle property.*

Proof. We prove the theorem using induction by the definition of superposition.

Induction basis: the graphs A, C and D have the triangle property.

Induction step: Let $S = G_{w \leftarrow H}$. By induction hypothesis the triangle property holds for graphs G and H.

Let x, y and z be internal nodes of the graph S. This means that each of them is an internal node of G or H and none of them is w. There are 8 options for choosing nodes x, y and z:

(1) $x, y, z \in H$. By induction hypothesis, the assertion holds.

(2) $x, y, z \in G$. We prove that if there exists a 0-path or a 1-path between the nodes y and z in the graph G, then such a path exists also in the graph $G_{w \leftarrow H}$. According to Theorem 3 there exist a 0-path $root(H) \rightsquigarrow low(w)$ and a 1-path $root(H) \rightsquigarrow high(w)$ in the graph $G_{w \leftarrow H}$. It is obvious that if some 0-path (1-path) goes through the node w, then this path goes also through the node $low(w)$ ($high(w)$). Therefore, if some 0-path $y \rightsquigarrow z$ goes through the node w (hence also through the node $low(w)$), then a 0-path $y \rightsquigarrow z$ exists also in the graph $G_{w \leftarrow H}$.

Similarly, if there exists a 1-path $z \rightsquigarrow y$ between the nodes z and y in the graph G, then a 1-path $z \rightsquigarrow y$ is also in the graph $G_{w \leftarrow H}$.

(3) $x, y \in H, z \in G$. Since there exists a 0-path $x \rightsquigarrow z$ in the graph $G_{w \leftarrow H}$, there also exists a 0-path $x \rightsquigarrow low(w)$. According to Theorem 3 there exists a 0-path to the node 0 from each internal node of H, therefore there exists a 0-path $y \rightsquigarrow low(w)$ in the graph S. Hence there exists a 0-path $y \rightsquigarrow z$ in S.

(4) $x \in H, y \in G, z \in H$. Dual to the case (3).

(5) $x \in H, y, z \in G$. According to the definition of superposition, the 1-path $x \rightsquigarrow y$ goes through the node $high(w)$ and the 0-path $x \rightsquigarrow z$ goes through the node $low(w)$. Therefore there exists a 1-path $w \rightsquigarrow y$ and a 0-path $w \rightsquigarrow z$ in the graph G. By induction hypothesis there exists a 1-path or a 0-path between y and z in $G \backslash \{w\}$ (because G is acyclic) and hence in S.

(6) $x \in G, y, z \in H$. The 1-path $x \rightsquigarrow y$ and 0-path $x \rightsquigarrow z$ must go through the unique root of H, which is impossible because superpositional graphs are homogenous.

(7) $x \in G, y \in H, z \in G$. Since there exists a 1-path $x \rightsquigarrow y$ in S, there exists a 1-path $x \rightsquigarrow w$ in G. Using the triangle property for G, there is either a 0-path $w \rightsquigarrow z$ or a 1-path $z \rightsquigarrow w$ in G. If there exists a 0-path $w \rightsquigarrow z$ (hence a 0-path $low(w) \rightsquigarrow z$) in the graph G and since there exists a 0-path $y \rightsquigarrow low(w)$ (by Theorem 3), then there also exists a 0-path $y \rightsquigarrow z$ in the graph $G_{w \leftarrow H}$.

If there exists a 1-path $z \rightsquigarrow w$ in the graph G, then there exist a 1-path $z \rightsquigarrow root(H)$ and a 1-path $root(H) \rightsquigarrow y$, which gives a 1-path $z \rightsquigarrow y$ in the graph $G_{w \leftarrow H}$.

(8) $x, y \in G, z \in H$. Dual to the case (7). □

It is easy to see that every superpositional graph is planar. We can prove even more restrictive property of superpositional graphs.

Definition 13. For internal nodes a and b of a binary traceable graph G we say that a *precedes* b ($a \prec b$) if a precedes b in the Hamiltonian path of the graph G. Also, for every internal node a, $a \prec 0$ and $a \prec 1$.

Definition 14. Edges $v_k \to v_p$ and $v_l \to v_r$ of a binary traceable graph are *crossing edges* if $v_k \prec v_l \prec v_p \prec v_r$.

Definition 15. We say that a binary traceable graph is *strongly planar* if it has no crossing 0-edges and no crossing 1-edges.

The strong planarity has a nice graph-theoretical interpretation: if we stretch a graph so that all nodes are in a straight line in the canonical order, then there are no 0-edges above the line and no 1-edges below the line. If a binary graph is strongly planar, then there are no crossing edges with the same label in such drawing. Figure 6 depicts a superpositional graph before and after stretching.

It is also obvious that if a binary graph is strongly planar, then it is also planar, while the opposite does not hold in general. In Figure 7 there is a binary graph, which is planar, but not strongly planar (as 1-edges are crossing).

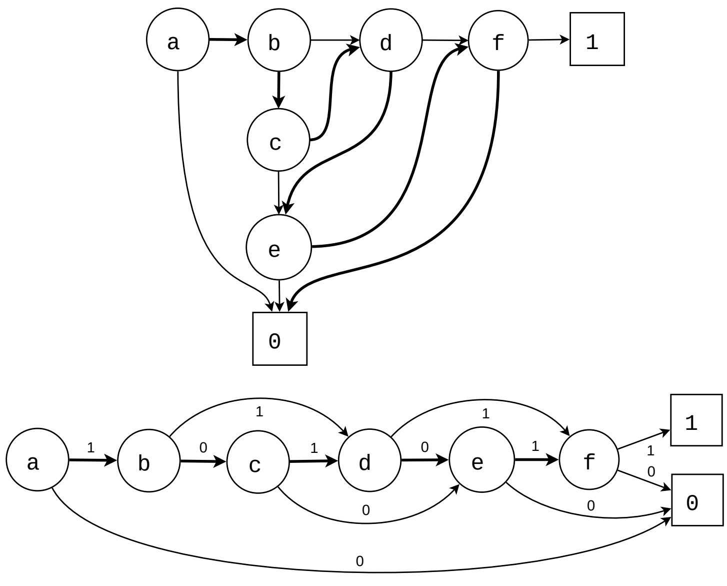

FIGURE 6. A superpositional graph before and after stretching.

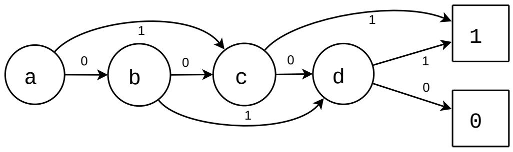

FIGURE 7. A planar binary graph, which is not strongly planar.

Theorem 7. *A binary traceable graph G is strongly planar if and only if G is homogenous and has the triangle property.*

Proof. We prove that

(1) if G is strongly planar, then the triangle property holds in the graph G,
(2) if G is strongly planar, then G is homogenous,
(3) if G is homogenous and has the triangle property, then G is strongly planar.

(1) Let G be strongly planar and v_i, v_j and v_k be three distinct internal nodes such that there exists a 1-path $v_i \rightsquigarrow v_j$ and a 0-path $v_i \rightsquigarrow v_k$. Let $k < j$. While moving forward from the node v_k using 1-edges, we will eventually be on the 1-path $v_i \rightsquigarrow v_j$ as the 1-edges do not cross. Thus there exists a 1-path $v_k \rightsquigarrow v_j$. If $k > j$, then there analogously exists a 0-path $v_j \rightsquigarrow v_k$.

(2) Let G be strongly planar. We assume by contradiction that both a 0-edge and a 1-edge enter into some internal node v_j. It is obvious that one of these edges is $v_{j-1} \rightarrow v_j$. Let it be the 0-edge (in the case of the 1-edge, the proof is similar). Now we show, that in this case, there exist two crossing 1-edges. Since the graph is binary, there exists a 1-edge $v_{j-1} \rightarrow v_m$, where $m > j$, but in this case the graph G is not strongly planar as this edge crosses the 1-edge coming into v_j (see Figure 8).

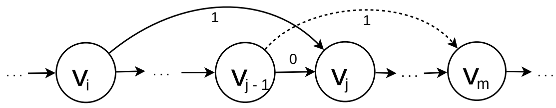

FIGURE 8. An illustrative figure for the proof of Theorem 7 (2).

(3) Let G be a traceable homogenous binary graph with the triangle property. We assume by contradiction that it is not strongly planar. Suppose that there are crossing 1-edges in G. Let $i < j < k < m$ be indexes so that 1-edges $v_i \rightarrow v_k$ and $v_j \rightarrow v_m$ are crossing edges, where j is minimal and i is maximal for that j, i.e., the choice of i depends on j (see Figure 9).

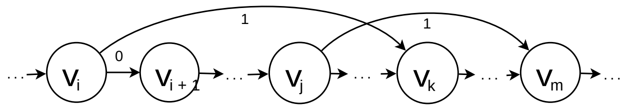

FIGURE 9. The edges $v_i \to v_k$ and $v_j \to v_m$ are crossing.

With this we guarantee that arbitrary 1-edge beginning between the nodes v_i and v_j does not cross the 1-edge $v_j \to v_m$. Due to the triangle property and acyclicity there is a 1-path $v_{i+1} \rightsquigarrow v_k$. This 1-path must contain v_j because of our choice of i, j, k and m. But then two 1-edges start from v_j, which contradicts the definition of a binary graph. □

Corollary 1. *Every superpositional graph is strongly planar.*

Proof. Immediate consequence of Theorems 4, 5, 6 and 7. □

3. Graph-theoretical description of superpositional graphs

The properties of superpositional graphs, proven in the previous section, are not sufficient for a binary graph to be a superpositional graph. There exists a binary traceable strongly planar graph (see Figure 10), which has all the properties, but can not be generated by any sequence of superpositions. Therefore we have to define additionally a more sophisticated property.

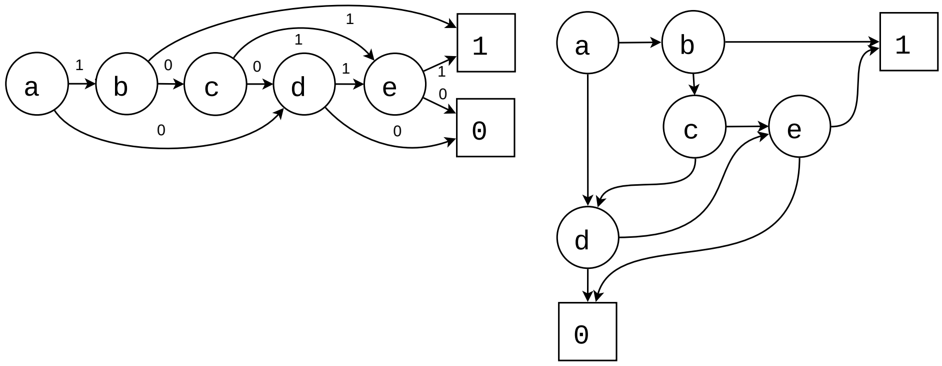

FIGURE 10. A binary traceable strongly planar graph, which is not a superpositional graph.

Definition 16. We say that a binary traceable graph is *1-cofinal* (*0-cofinal*) if all 1-edges (0-edges), starting between the endpoints of some 0-edge (1-edge) and crossing it, are entering into the same node.

Figure 11 and Figure 12 are illustrating these notions.

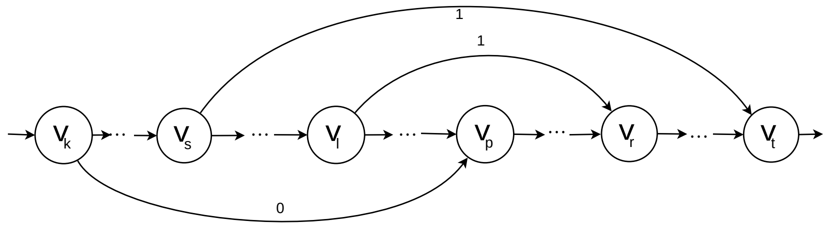

FIGURE 11. The situation, forbidden by 1-cofinality.

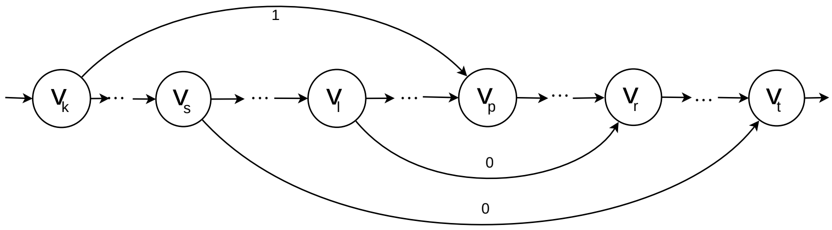

FIGURE 12. The situation, forbidden by 0-cofinality.

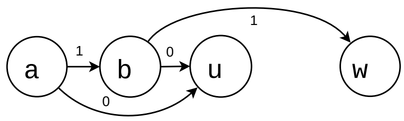

FIGURE 13. A fragment of a graph with crossing 0-edge and 1-edge.

Definition 17. A binary traceable graph is *cofinal* if it is 1-cofinal and 0-cofinal.

Theorem 8. *Every superpositional graph is cofinal.*

Proof. We prove the 1-cofinality, the proof of 0-cofinality is similar. For the verification of 1-cofinality, it is enough to show that if there is a pair of 1-edges, starting between the endpoints of some 0-edge and crossing it, then these 1-edges must have a common endpoint. Let us suppose that we have a 1-cofinal superpositional graph G and a 0-edge $a \to u$ in it. The only way to obtain, applying superposition, two 1-edges that start between a and u and cross $a \to u$ is to superpose a graph H instead of some b between a and u, which is a starting point of some 1-edge, crossing $a \to u$ (see Figure 13). All internal edges of H remain unchanged. The only new edges, which will cross the 0-edge, are the 1-edges, which were pointing to the terminal 1 of H, but in $G_{b \leftarrow H}$ were redirected to $w = high(b)$. If graphs G and H

are 1-cofinal, then so is $G_{b\leftarrow H}$. This means that 1-cofinality is preserved by superposition. □

Lemma 1. *If G is a traceable strongly planar cofinal binary graph with $n > 2$ internal nodes, then it can be represented by a superposition $G = H_{w\leftarrow F}$, where H and F are binary graphs with at least 2 internal nodes.*

Proof. Let $v_1, \ldots, v_n$ be the canonical sequence of internal nodes of the graph G. We are looking for a subsequence $v_k, v_{k+1}, \ldots, v_l$ $(k < l)$ such that:

(1) All incoming edges from the nodes $v_1, \ldots, v_{k-1}$ to the nodes of the subsequence are pointing to v_k.
(2) All 1-edges from the nodes of the subsequence to the nodes $v_{l+1}, \ldots, v_n, 1$ are pointing to the same node.
(3) All 0-edges from the nodes of the subsequence to the nodes $v_{l+1}, \ldots, v_n, 0$ are pointing to the same node.

We construct binary graphs H and F, using the subsequence. The set of nodes of the graph H is $V(H) = \{v_1, \ldots, v_{k-1}, w, v_{l+1}, \ldots, v_n, 0, 1\}$. The edges of H are all edges of G with both endpoints in $V(H)$; all edges, pointing to v_k will be redirected to the node w and the 1-edges (0-edges) from some node of the subsequence to a node from the set $\{v_{l+1}, \ldots, v_n, 0, 1\}$ will be replaced by a 1-edge (0-edge) in H from w to the same node. Formally,

$$\begin{aligned} E(H) \quad = \quad & \{(u, v): \; u, v \in V(H) \setminus \{w\}, (u, v) \in E(G)\} \cup \\ & \{(u, w): \; u \in \{v_1, \ldots, v_{k-1}\}, (u, v_k) \in E(G)\} \cup \\ & \{(w, z): \; z \in \{v_{l+1}, \ldots, v_n, 0, 1\}, \exists i_{k\le i\le l}((v_i, z) \in E(G))\}. \end{aligned}$$

The set of nodes of the graph F is $V(F) = \{v_k, \ldots, v_l, 0, 1\}$. The edges of F are all edges of G with endpoints in $V(F)$; all 1-edges of G, going from $V(F)$ to the nodes from $\{v_{l+1}, \ldots, v_n, 1\}$ will be redirected to the terminal 1 and all 0-edges of G, going from $V(F)$ to the nodes from $\{v_{l+1}, \ldots, v_n, 0\}$, will be redirected to the terminal 0 of F. Formally,

$$\begin{aligned} E(F) \quad = \quad & \{(u, v): \; u, v \in V(F), (u, v) \in E(G)\} \cup \\ & \{(v_i, 1): \; v_i \in V(F), \exists z_{z\in\{v_{l+1},\ldots,v_n,1\}}(high(v_i) = z)\} \cup \\ & \{(v_i, 0): \; v_i \in V(F), \exists z_{z\in\{v_{l+1},\ldots,v_n,0\}}(low(v_i) = z)\}. \end{aligned}$$

Obviously, the construction of graphs H and F in conditions (1)–(3) is a reverse engineering of the notion of superposition of binary graphs. Therefore $G = H_{w\leftarrow F}$. It is left to show that on the premises of the lemma, there always exists a subsequence of the canonical sequence of nodes of the graph G, which satisfies conditions (1)–(3). We have to consider four cases:

1) $high(v_1) = 1$. Then $low(v_1) = v_2$ and the sequence is $v_2, \ldots, v_n$.

2) $low(v_1) = 0$. Then $high(v_1) = v_2$ and the sequence is $v_2, \ldots, v_n$.

3) $high(v_1) = v_{l+1}$, where $1 < l \le n - 1$. Then $low(v_1) = v_2$ and the sequence is $v_1, \ldots, v_l$. Condition (1) is fulfilled, because v_1 is the first node

of the canonical sequence of nodes. Condition (2) is fulfilled, because if there is some 1-edge (v_i, z), where $2 \le i \le l, z \in \{v_{l+2}, \ldots, v_n, 1\}$, then we have two crossing 1-edges, and G is not strongly planar. Condition (3) is fulfilled, because if there are two 0-edges, (v_i, u) and (v_j, z), where $2 \le i < j \le l$ and $u, z \in \{v_{l+2}, \ldots, v_n, 0\},\ u \ne z$, then G is not cofinal.

4) $low(v_1) = v_{l+1}$, where $1 < l \le n - 1$. Then $high(v_1) = v_2$ and the sequence is $v_1, \ldots, v_l$. Conditions (1)–(3) are fulfilled by considerations, symmetrical to the case 3). □

Theorem 9. *A binary graph is a superpositional graph if and only if it is a strongly planar cofinal traceable graph.*

Proof. $\Longrightarrow$ Proved by Theorem 5, Corollary 7 and Theorem 8.

$\Longleftarrow$ We prove the claim by induction over the number of internal nodes of the graph.

Induction basis. There are two binary graphs with 2 internal nodes. These are binary graphs C and D (Figure 1), which are superpositional graphs.

Induction step. Let G be a strongly planar cofinal traceable graph with n internal nodes ($n > 2$). By Lemma 1, it can be represented as a superposition of two graphs: $G = H_{w \leftarrow F}$, where $|V(H)| < |V(G)|$ and $|V(F)| < |V(G)|$. The graphs H and F inherited from G the properties of traceability, strong planarity and cofinality. By induction hypothesis H and F are superpositional graphs. Graph G is a superposition of two superpositional graphs and by Theorem 2 is a superpositional graph. □

As an illustration of an application of Theorem 9, we reproduce binary graphs from Figure 3, stretched by the canonical ordering of nodes. The graph, leftmost in Figure 3, is depicted in Figure 14 and the rightmost graph in Figure 15.

It is obvious, that both of them are traceable and strongly planar. The graph in Figure 14 is not cofinal. Nodes v and w are between the endpoints of 1-edge $u \to x$, but 0-edges $v \to z$ and $w \to y$ are pointing to different nodes. According to Theorem 9, the graph is not a superpositional graph. The graph in Figure 15 is cofinal and therefore it is a superpositional graph.

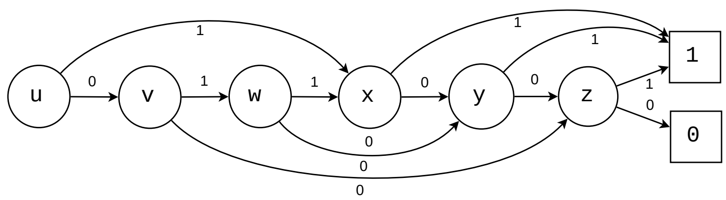

FIGURE 14. Leftmost graph (from Figure 3) in canonical order.

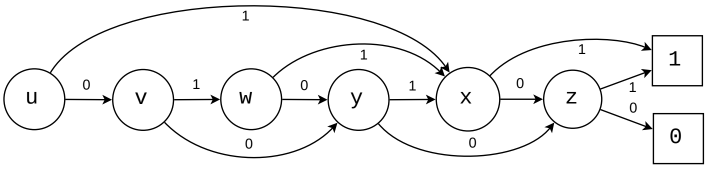

FIGURE 15. Rightmost graph (from Figure 3) in canonical order.

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