**From the matrix perturbation theory:**

Let us solve the system

$$Ax = \mathbf{b} \tag{8}$$

where $A \in \mathbb{R}^{n\times n}$ and $\mathbf{b} \in \mathbb{R}^n$ are given and $\mathbf{x} \in \mathbb{R}^n$ is unknown. Suppose, $A$ is given with an error $\tilde{A} = A + \delta A$. Perturbed solution satisfies the system of linear equations

$$(A + \delta A)(\mathbf{x} + \delta\mathbf{x}) = \mathbf{b}. \tag{9}$$

**Theorem 5.1.** Let $A$ be nonsingular and $\delta A$ be sufficiently small, such that

$$\|\delta A\|_\infty \|A^{-1}\|_\infty \leq \frac{1}{2}. \tag{10}$$

Then $(A + \delta A)$ is nonsingular and

$$\frac{\|\delta\mathbf{x}\|_\infty}{\|\mathbf{x}\|_\infty} \leq 2\kappa(A)\frac{\|\delta A\|_\infty}{\|A\|_\infty}, \tag{11}$$

where $\kappa(A) = \|A\|_\infty \|A^{-1}\|_\infty$ is the condition number.

***Proof of the theorem.***

Subtracting (8) from the system of equations (9) we get $(A + \delta A)\delta\mathbf{x} = -\delta A\ \mathbf{x}$. Multiplying it with expression $(A+\delta A)^{-1}$ and taking norms, we get

$$\|\delta\mathbf{x}\|_\infty = \| - (A + \delta A)^{-1}\ \delta A\ \mathbf{x}\|_\infty \leq \left\{\|(A + \delta A)^{-1}\|_\infty \|A\|_\infty\right\} \frac{\|\delta A\|_\infty}{\|A\|_\infty}\|\mathbf{x}\|_\infty\ .$$

To end the proof we need to show that the expression in the brackets $\{\cdot\}$ can be estimated with expression $2\kappa(A)$, or equivalently,

$$\|(A + \delta A)^{-1}\|_\infty \leq 2\|A^{-1}\|_\infty\ . \tag{12}$$

To check the validity of estimation (12) we notice first, that if $X$ is an arbitrary $n \times n$ matrix with real values satisfying the criteria $\|X\|_\infty < 1$, then $\|X^n\|_\infty \leq \|X\|_\infty^n \to 0 \text{ with } n \to \infty$. It follows that,

$$(I - X)(I + X + X^2 + \ldots + X^n) = I - X^{n+1} \to I, \text{ with } n \to \infty\ .$$

Therefore $(I - X)^{-1} = \sum_{j=0}^{\infty} X^j$ and

$$\|(I - X)^{-1}\|_\infty \leq \sum_{j=0}^{\infty} \|X^j\|_\infty \leq \sum_{j=0}^{\infty} \|X\|_\infty^j = (1 - \|X\|_\infty)^{-1}\ . \tag{13}$$

We can write

$$(A + \delta A) = \left[ I + \delta A \ A^{-1} \right] A. \tag{14}$$

Let's take $X = -\delta A \ A^{-1}$. Assuming (10), $\|X\|_\infty \leq 1/2 < 1$ we can apply (13) to show that $(I + \delta A \ A^{-1})$ is nonsingular and

$$\|(I + \delta A \ A^{-1})^{-1}\|_\infty \leq (1 - \| - \delta A \ A^{-1}\|_\infty)^{-1} \leq 2 \ .$$

Therefore (14) gives that $A + \partial A$ nonsingular and

$$\|(A + \delta A)^{-1}\|_\infty = \|A^{-1}(I + \delta A \ A^{-1})^{-1}\|_\infty \leq 2\|A^{-1}\|_\infty$$

and (12) follows.