Notes on Feynman Parametrisation and the Dirac Delta Function

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The Feynman parametrisation is a way to write fractions with a product in the denominator:

$$
\begin{aligned}
&\frac{1}{A_1 A_2 \ldots A_m} = \\
&(m-1)! \int_0^1 du_1 \int_0^1 du_2 \ldots \int_0^1 du_m \frac{\delta(1 - u_1 - \ldots - u_m)}{[A_1 u_1 + A_2 u_2 + \ldots + A_m u_m]^m},
\end{aligned} \tag{1}
$$

invented by Richard Feynman to calculate loop integrals.

1 Delta Function and Integration Limits

In Feynman parametrisation, we have a multiple definite integral of the form

$$
\int_0^1 du_1 \int_0^1 du_2 \ldots \int_0^1 du_m \delta(1 - u_1 - \ldots - u_m) f(u_1, \ldots, u_m). \tag{2}
$$

Integrating over u_m, the delta function sets u_m to $1 - u_1 - \ldots - u_{m-1}$ in f, but not only that. The delta function depends on the parameters $u_1, \ldots, u_{m-1}$. Should the peak of the delta function fall outside of the integration interval $[0, 1]$, the integral over u_m is zero; otherwise it is one.[1] This is expressed by

$$
\int_0^1 du_m \delta(1 - u_1 - \ldots - u_m) = \theta(1 - u_1 - \ldots - u_{m-1}))\theta(u_1 + \ldots + u_{m-1}). \tag{3}
$$

The second unit step is equivalent to

$$
0 \leqslant u_1 + \ldots + u_{m-1}, \tag{4}
$$

which condition is automatically fulfilled in subsequent integrations. The first unit step says that

$$
u_1 + \ldots + u_{m-1} \leqslant 1, \tag{5}
$$

[1] Note that for this to be true, the integral over the delta function must equal one even when the argument of the delta function coincides with one of the integration limits. (For some other applications, it is more convenient to define the value of the integral as $1/2$ in that case.) The same carries over to the unit step function.

effectively setting the upper limit for the next integral (over u_{m-1}) to $1-u_1-\ldots-u_{m-2}$, and so on.

For example

$$\begin{aligned} &\int_0^1 du_1 \int_0^1 du_2 \int_0^1 du_3 \delta(1-u_1-u_2-u_3) f(u_1,u_2,u_3) = \\ &= \int_0^1 du_1 \int_0^{1-u_1} du_2 f(u_1,u_2,1-u_1-u_2). \end{aligned} \tag{6}$$

2 Proof of a Form of Feynman Parametrization

A variation of Feynman parametrization is

$$\begin{aligned} &\frac{1}{A_1 A_2 \ldots A_m} = \\ &\frac{(m-1)! \int_0^1 du_1 \int_0^1 du_2 \ldots \int_0^1 du_{m-1} \quad u_1^{m-2} \ldots u_{m-2}}{[A_m u_1 \ldots u_{m-1} + A_{m-1} u_1 \ldots u_{m-2}(1-u_{m-1}) + \ldots + A_1(1-u_1)]^m} \end{aligned} \tag{7}$$

that may be easier to integrate than the usual form, as all the integration limits are the same.

It is easy to prove (7) directly by induction. For two factors in the numerator, the parametrization is

$$\begin{aligned} &(2-1)! \int_0^1 du \frac{1}{[Bu + A(1-u)]^2} \\ &= \int_0^1 du \frac{1}{[(B-A)u + A]^2} \\ &= -\frac{1}{B-A}\left[\frac{1}{(B-A)+A} - \frac{1}{A}\right] \\ &= \frac{1}{A-B}\left[\frac{1}{B} - \frac{1}{A}\right] \\ &= \frac{1}{AB}. \end{aligned} \tag{8}$$

Assuming (7) for m factors in the denominator, we must show that for $m+1$ factors, after taking the integral over u_m, we get the product of $1/A_{m+1}$ with (7).

We have

$$\begin{aligned} &\frac{m! \int_0^1 du_1 \int_0^1 du_2 \ldots \int_0^1 du_m \quad u_1^{m-1} \ldots u_{m-1}}{[A_{m+1} u_1 \ldots u_m + A_m u_1 \ldots u_{m-1}(1-u_m) + \ldots + A_1(1-u_1)]^{m+1}}. \end{aligned} \tag{9}$$

The integration variable u_m appears only in the first two terms. Regrouping

$$
\begin{aligned}
& m! \int_0^1 du_1 \int_0^1 du_2 \ldots \int_0^1 du_m \\
& \frac{u_1^{m-1} \ldots u_{m-1}}{[(A_{m+1} - A_m) u_1 \ldots u_m + A_m u_1 \ldots u_{m-1} + \ldots + A_1(1 - u_1)]^{m+1}} \\
& = -\frac{m!}{m} \frac{1}{A_{m+1} - A_m} \int_0^1 du_1 \int_0^1 du_2 \ldots \int_0^1 du_{m-1} \frac{u_1^{m-1} \ldots u_{m-1}}{u_1 \ldots u_{m-1}} \\
& \big\{ [(A_{m+1} - A_m) u_1 \ldots u_{m-1} + A_m u_1 \ldots u_{m-1} + \ldots + A_1(1 - u_1)]^{-m} \\
& - [A_m u_1 \ldots u_{m-1} + \ldots + A_1(1 - u_1)]^{-m} \big\} \\
& = (m-1)! \frac{1}{A_m - A_{m+1}} \int_0^1 du_1 \int_0^1 du_2 \ldots \int_0^1 du_{m-1} u_1^{m-2} \ldots u_{m-2} \\
& \big\{ [A_{m+1} u_1 \ldots u_{m-1} + A_{m-1} u_1 \ldots (1 - u_{m-1}) + \ldots + A_1(1 - u_1)]^{-m} \\
& - [A_m u_1 \ldots u_{m-1} + A_{m-1} u_1 \ldots (1 - u_{m-1}) + \ldots + A_1(1 - u_1)]^{-m} \big\} \\
& = \frac{1}{A_m - A_{m+1}} \left(\frac{1}{A_{m+1} A_{m-1} \ldots A_1} - \frac{1}{A_m A_{m-1} \ldots A_1} \right) \\
& = \frac{1}{A_{m-1} \ldots A_1} \frac{1}{A_m - A_{m+1}} \left(\frac{1}{A_{m+1}} - \frac{1}{A_m} \right) \\
& = \frac{1}{A_1 A_2 \ldots A_m},
\end{aligned}
\tag{10}
$$

where we have taken the integrals with the help of (7).